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MATCHING STUDENTS TO SCHOOLS

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In this paper, we present the problem of matching students to schools by using different matching mechanisms. This market is specific since public schools are free and the price mechanism cannot be used to determine the optimal allocation of children in schools. Therefore, it is necessary to use different matching algorithms that mimic the market mechanism and enable us to determine the core of the cooperative game. In this paper, we will determine that it is possible to apply cooperative game theory in matching problems. This review paper is based on illustrative examples aiming to compare matching algorithms in terms of the incentive compatibility, stability and efficiency of the matching. In this paper we will present some specific problems that may occur in matching, such as improving the quality of schools, favoring minority students, the limited length of the list of preferences and generating strict priorities from weak priorities.

Keywords: matching, Boston algorithm, deferred acceptance algorithm, top trading cycles

JEL Classification: C78

INTRODUCTION

The most important role of market prices is to determine the optimal allocation of resources. However, there are certain situations where the price mechanism cannot be applied, but it is still necessary to optimally allocate resources. One example is the enrolment of students in public high schools that are free. Therefore, it is necessary to construct an algorithm for matching students to schools that mimics the functioning of the market. The subject of this research is matching students to schools in the absence of money transfers, while the objective of the research is to show the possibility for the practical application of matching algorithms in this market.

From the methodological point of view, matching algorithms are based on the application of cooperative game theory and the mechanism design.

The basic division of games is that into cooperative and non-cooperative games. In non-cooperative games, the player maximizes his payoff for a given set of rules of the game (Backović & Popović, 2012). In cooperative games, players make coalitions and the strategy is defined at the level of the coalition (Backović, Popović & Stamenković, 2016). In cooperative games, it is necessary to determine the

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core of the game, when there is no coalition of players that can improve upon the allocation in the core.

The mechanism design is based on the assumption that players possess private information that they submit to the center of the mechanism. The mechanism should be based on the rules such that each player finds optimal to reveal his private information. This mechanism is incentive-compatible.

The first hypothesis which matching algorithms are based on is that it is possible to determine the core of the game and the allocation that would be obtained in the market mechanism with money transfers. The second hypothesis is that it is possible to create incentive-compatible matching algorithms.

This paper is a review paper that employs original examples to illustrate the existing results in a concise way.

In the selection of the matching mechanism, it is necessary to take into account the fact that students have an incentive to express their true preferences and that matching is stable and efficient.

A. Roth (2015) states that the matching mechanism used in New York until 2003 was very complex. The algorithm of immediate matching was used at that time, operating as follows. Students submit a list of three schools they want to attend. On the basis of the received applications, the school enrolls students with the highest priorities, who have listed that school as the first choice. If the capacity of the school is filled in the first round, it declines other applicants. In the second round, the students who were previously rejected apply to the school which is their second best choice. Schools with empty seats enroll applicants with the highest priority up to their capacity and reject other applicants. The same procedure is applied in the third round. Given the characteristics of the algorithm, it was not necessary for students to submit their preferences for more than three schools since the likelihood of their being enrolled in a school that was their fourth best choice was negligible. The students who were not enrolled after the third round are administratively assigned in the schools where there were vacancies. This algorithm was very inefficient since almost 30% of the students were enrolled in this manner. Based on the stated preferences, it can be determined that about 80% of the students were enrolled in their first-choice schools. However, the students in this algorithm do not have an incentive to express their true preferences and the school listed as the best choice in the most cases represents a strategic choice. In other words, the school listed as the best choice represents the school in which the student estimates the probability of being admitted as the highest, and often this school was not his real first choice. A more detailed analysis of the immediate matching algorithm is given in: A. Abdulkardiroglu, A. Pathak and A. Roth (2005; 2009); A. Abdulkardiroglu, A. Pathak, A. Roth and T. Sonmez (2005). Given all these problems, Alvin Roth and his colleagues have proposed the application of the deferred acceptance algorithm, which performs a temporary matching. For the application of this algorithm, A. Roth was awarded the Nobel Prize in economics in 2012, together with L. Shapely.

The deferred acceptance algorithm was first considered by D. Gale and L. Shapely (1962). Specifying the list of preferences in the deferred acceptance algorithm is much simpler. The student does not have to consider in which school there is the highest probability for him to enroll because he will not lose a priority in the school against the other students with a lower priority, as is the case in the immediate matching algorithm. Thus, the student submits his true preferences in this algorithm. This mechanism leads to a stable matching, which means that it is not possible to find a school and a student not paired with each other and that prefer each other to the pair assigned to them in the algorithm. In other words, in a stable allocation, there is no justified envy. This algorithm has proven to be significantly more successful than the immediate matching algorithm and the number of the administratively enrolled students in New York fell by ten times. At the same time, there were more students enrolled in their first choice schools, as well as more students enrolled in their second choice schools, and so on. The deferred acceptance algorithm was selected although the top trading cycle introduced by D. Gale and H. Scarf (1974) was also considered. In this mechanism, students also submit their true preferences.

School priorities are determined based on the distance of the student's residence from the school and on the basis of whether a student has a sibling already attending that school. The students who live closer to the school have a higher priority, which is also the case with the students whose siblings have already been enrolled in that school. Priorities can be determined exogenously, when they are submitted to schools by an administrative entity, which is the case in Boston. In this case, there is a one-sided matching because students' preferences are more important than exogenously determined priorities. Another option implies that such priorities are determined by schools, which is the case in New York. In this case, priorities can also be viewed as preferences, which leads us to a two-sided matching problem. In two-sided matching, schools' preferences are equally important as students' preferences, as opposed to one-sided matching, where only students' preferences are relevant. When the problem of one-sided matching is concerned, no stable allocation may be Pareto optimal, unlike two-sided matching, where there is no difference between these two objectives. As far as incentive compatibility, or the true revelation of preferences, are concerned, A. Abdulkadiroğlu (2013) claims that the algorithm with this feature greatly facilitates the student's decision on submitting his preference list.

The rest of the paper is organized as follows. In the second part, the Boston matching algorithm, the deferred acceptance algorithm and the top trading cycle algorithm are presented. In the third part, a more detailed analysis of these mechanisms follows, from the point of view of incentive compatibility, stability and efficiency. The fourth part considers whether matching algorithms respect the improved ranking of the school on the preference list due to its increased quality, as well as its enrollment policy favoring minority students. In the fifth part, the properties of the matching algorithms are considered in the case of the limited length of the preference list. In the sixth part, the methods for constructing strict priorities from weak priorities are considered. The last section is reserved for the concluding remarks.

MATCHING ALGHORITMS

Matching algorithms can be illustrated by way of the following example. Suppose there are four schools (c_1, c_2, c_3, c_4) with one place and four students (s_1, s_2, s_3, s_4). The preferences and priorities are shown in Table 1 and Table 2.

Table 1 Priorities

$\succ_{c_1}$	$\succ_{c_2}$	$\succ_{c_3}$	$\succ_{c_4}$
s_4	s_1	s_4	s_2
s_2	s_2	s_3	s_3
s_3	s_3	s_2	s_1
s_1	s_4	s_1	s_4

Source: Author

Table 2 Preferences

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$	$\succ_{s_4}$
c_1	c_4	c_1	c_4
c_2	c_3	c_2	c_2
c_3	c_2	c_4	c_1
c_4	c_1	c_3	c_3

Source: Author

The algorithm used in Boston is presented first, and we will see its main drawback. In the Boston algorithm, matching is immediate. Each student applies to the school that is his best choice. Schools keep students with the highest priority and reject others. In the next step, the rejected students apply to the second choice school and schools retain the students with the highest priority. The process is repeated until all students are matched to schools.

In the previous example, in the first step, Students 1 and 3 apply to School 1, whereas Students 2 and 4 apply to School 4 (Table 3).

Table 3 Boston algorithm (1)

c_1	c_2	c_3	c_4
s_1, s_3			s_2, s_4

Source: Author

School 1 retains Student 3, who has a higher priority, while School 4 retains Student 2. In the next step, Students 1 and 4 apply to School 2 (Table 4).

Table 4 Boston algorithm (2)

c_1	c_2	c_3	c_4
s_3			s_2
	s_1,s_4		

Source: Author

School 2 retains Student 1, who has a higher priority. In the next step, Student 4 applies to School 1, but there are no empty seats. In the last step, this student applies to School 3, which he is paired with. Thus, in the Boston algorithm, the following matching occurs (Table 5).

Table 5 Boston algorithm (3)

c_1	c_2	c_3	c_4
s_3	s_1	s_4	s_2

Source: Author

In the deferred acceptance algorithm, students and schools are temporarily matched and the school may reject the students with whom it is temporarily matched in favor of the students with a higher priority, who apply later. In the first step, there is the same situation as in the Boston algorithm (Table 6).

Table 6 Deferred acceptance algorithm (1)

c_1	c_2	c_3	c_4
s_1,s_3			s_2, s_4

Source: Author

The same situation repeats in the second step, when Students 1 and 4 apply to School 2 (Table 7).

Table 7 Deferred acceptance algorithm (2)

c_1	c_2	c_3	c_4
s_3			s_2
	s_1, s_4		

Source: Author

In the third step, Student 4 applies to School 1 (Table 8).

Table 8 Deferred acceptance algorithm (3)

c_1	c_2	c_3	c_4
s_3			s_2
	s_1		
s_4			

Source: Author

Now, School 1 keeps Student 4 as the best option and rejects Student 3, who applies in the next step in School 2 (Table 9).

Table 9 Deferred acceptance algorithm (4)

c_1	c_2	c_3	c_4
			s_2
	s_1		
s_4			
	s_3		

Source: Author

School 2 retains Student 1 as the best option and in the last step, Student 3 applies to School 3, which he is paired with (Table 10).

Table 10 Deferred acceptance algorithm (5)

c_1	c_2	c_3	c_4
s_4	s_1	s_3	s_2

Source: Author

In the top trading cycle algorithm, each student points to his first choice school and each school points to the student with the highest priority. The cycle starts with the student *i*, who points to the school *k* that points to the student *j*, etc., wherein the last school points to the student *i*, who has actually started the cycle. The students in the cycle are matched to the schools they point to and are removed from the algorithm. The process repeats until all students have been matched.

In the first step, the situation is as follows (Figure 1):

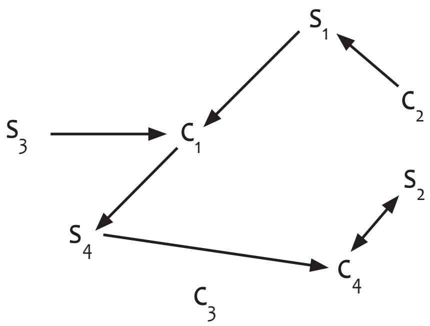

Figure 1 Top trading cycle (1)

Source: Author

Based on Figure 1, we can determine that there is a cycle consisting of School 4 and Student 2, matched and removed from the algorithm.

In the second step, the students and the schools point to their best option among the remaining schools and students (Figure 2).

Based on Figure 2, we can determine that there is a cycle consisting of (s_1, c_1, s_4, c_2). Thus, Student 1 is matched to School 1, whereas Student 4 is matched to School 2. In the last step, one cycle remains and School 3 is matched to Student 3 (Figure 3).

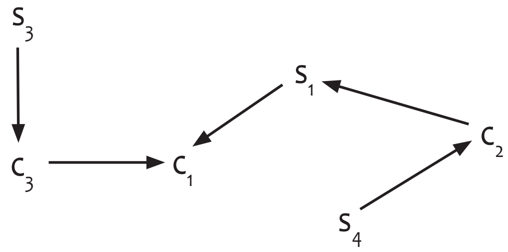

Figure 2 Top trading cycle (2)

Source: Author

Figure 3 Top trading cycle (3)

Source: Author

Therefore, in the top trading cycle algorithm, the allocation is as follows (Table 11):

Table 11 Top trading cycle

c_1	c_2	c_3	c_4
s_1	s_4	s_3	s_2

Source: Author

If the top trading cycle allocation and the allocation in the deferred acceptance algorithm are compared to each other, we can see that in the first allocation, there is justified envy when there are the student *i* and the school *j*, such that the student *i* prefers the school *j* to the school he is assigned in in the algorithm, whereas in the school *j* the student *i* has a higher priority than the student *l*, who is matched to the school *j* in the algorithm. If there is justified envy, matching is not stable. Specifically, Student 3 prefers School 1, which Student 1 is enrolled in, whereas Student 3 has a higher priority in School 1 than Student 1. On the other hand, we can see that the top trading cycle allocation is more efficient than the allocation in the deferred acceptance algorithm, since Students 1 and 4 are matched to the schools that have a higher rank on their preference lists, and Students 2 and 3 are indifferent between the two allocations.

PROPERTIES OF MATCHING ALGHORITMS

In the previous example, we have started from the assumption that the students in each algorithm express their true preferences. However, the problem with the Boston algorithm is that students have an incentive to misrepresent their preferences and the student lists the school which he estimates that he has the highest likelihood to enroll in as the best choice although it may not be his real best choice. This means that in the Boston mechanism the Nash equilibrium must be determined. This can be illustrated by the example of two students and two schools with only one place (Table 12). Suppose that the payoff is 2 if the student is enrolled in the preferred school; otherwise, his payoff is 1.

Table 12 Preferences and priorities

$\succ_{s_1}$	$\succ_{s_2}$		$\succ_{c_1}$	$\succ_{c_2}$
c_1	c_2		s_2	s_2
c_2	c_1		s_1	s_1

Source: Author

Each student has two strategies at his disposal. The first strategy is to truthfully submit his preferences and the second is to revise the order of his actual preferences. In its normal form, this game has two Nash equilibria (Table 13).

Table 13 Nash equilibria

		Student 2	
		c_1, c_2	c_2, c_1
Student 1	c_1, c_2	1 1	2 2
	c_2, c_1	1 1	2 2

Source: Author

In the first Nash equilibrium, both students truthfully reveal their preferences, whereas in the second Nash equilibrium, Student 1 has changed the order of his real preferences. A more detailed consideration of the Nash equilibrium in the Boston algorithm can be found in P. Pathak and T. Sönmez (2008).

The previous game that we have considered is a static game with perfect information. If players only know the probability distribution of the possible types of the other players, where the type of the player represents the order of his preferences, but do not know with certainty their preferences, the game is with imperfect information. H. Ergin and T. Sönmez (2006) argue that, in the game of imperfect information, students may be better off in the Boston algorithm than in the deferred acceptance algorithm.

In the previous discussion, we have seen that the main disadvantage of the Boston algorithm is that students do not express their true preferences. However, the Boston algorithm has some desirable properties according to F. Kojima and U. Ünver (2014). First, this algorithm strictly respects the stated order of preferences, which means that if a student is not matched to a school he prefers to the school which he is matched to, the preferred school has filled the places with the students who have listed that school in a higher place on their preference lists. Another advantage of the Boston algorithm is that an increase in the number of available places in schools may not make students worse off. The third advantage of this mechanism is that, if the number of the students who participate in matching decreases, the other students may not be in a worse position. Finally, if the student matched to a particular school is removed from the matching process, the assignment of the other students will remain the same.

The deferred acceptance algorithm leads to a true preference revelation. Another advantage of this algorithm is the elimination of justified envy, whereas the most serious drawback is that such an allocation is not efficient. In one-to-one matching, where the one side of the market makes a proposal to the other side, agents on the proposing side have an incentive to reveal their true preferences. However, the matching of students to schools is the case of many-to-one matching because schools can be matched to more students and each student can be matched to only one school. A. Roth (1985) has proved that, in the algorithm in which students make a proposal to schools, students reveal their true preferences. However, in the algorithm in which schools make a proposal to students, schools have an incentive to misrepresent their priorities. This can be illustrated by the following example of 3 schools and 4 students, where the first school can enroll two students and the second and third schools can only enroll one student. Suppose that the students and the schools have the following preferences and priorities (Table 14 and Table 15).

Table 14 Priorities

$\succ_{c_1}$	$\succ_{c_2}$	$\succ_{c_3}$
s_1	s_1	s_3
s_2	s_2	s_1
s_3	s_3	s_2
s_4	s_4	s_4
$q_1=2$	$q_2=1$	$q_3=1$

Source: Roth, 1985

Table 15 Preferences

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$	$\succ_{s_4}$
c_3	c_2	c_1	c_1
c_1	c_1	c_3	c_2
c_2	c_3	c_2	c_3

Source: Roth, 1985

First, the schools are assumed to submit their true priorities. In this case, by applying the deferred acceptance algorithm, where the school makes a proposal, the following matching is obtained: $\mu^C = [(c_1, (s_3, s_4)), (c_2, s_2)\ (c_3, s_1)]$. However, School 1 can be better off by misrepresenting its priorities. Suppose School 1 omits Students 1 and 3 from the list of its priorities and submits the priority: $\succ_{c_1}'$: s_2, s_4. By applying the deferred acceptance algorithm, in which schools make a proposal, the following matching is obtained: $\mu^{C'} = [(c_1, (s_2, s_4)), (c_2, s_1)\ (c_3, s_3)]$. School 1 is better off with this strategy, because it is matched to Students 2 and 4 and when it submits true priorities, it is matched to Students 3 and 4.

In addition to the manipulation of priorities, schools can manipulate the capacity in order to be paired with a set of preferred students. The school cannot report a higher capacity than the real one is, but it can report that it has a smaller capacity. This problem has been extensively analyzed by T. Sönmez (1997).

Let us now consider the following example, in which there are 3 students and 2 schools, where School 1 can enroll two students, whereas School 2 can only enroll one student. The preferences and priorities are accounted for in Table 16 and Table 17.

Table 16 Priorities

$\succ_{c_1}$	$\succ_{c_2}$
s_1	s_3
s_2, s_3	s_1
s_2	s_2
s_3	
$q_1=2$	$q_2=1$

Source: Author

Table 17 Preferences

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$
c_2	c_1	c_1
c_1	c_2	c_2

Source: Author

If both schools report the true capacity, the deferred acceptance algorithm, in which students make a proposal, leads to the matching: $\mu^S (q_1 = 2, q_2 = 1) = ((c_1, (s_2, s_3)), (c_2, s_1))$.

Suppose School 1 reports that it has a lower capacity and that it can only enroll one student. After this manipulation, the deferred acceptance algorithm leads to the allocation: $\mu^S (q_1 = 1, q_2 = 1) = ((c_1, s_1), (c_2, s_3))$. We can see that the school is better off with such a manipulation because it is matched to Student 1, preferred to Students 2 and 3. Somehow paradoxically, School 2 is in a better position due to the capacity manipulation by School 1 since it is now paired with Student 3, who is preferred to Student 1.

We have seen that schools have an incentive to misrepresent their priorities and capacities. However, F. Kojima and P. Pathak (2009) argue that in large markets incentives for these two types of manipulation tend to zero.

When the top trading cycle algorithm is concerned, it leads to true preference revelation and an efficient allocation. The biggest drawback of this algorithm is that it does not eliminate justified envy. A more detailed comparison of the characteristics of these two algorithms can be found in: A. Abdulkardiroğlu and T. Sönmez (2003); A. Abdulkardiroğlu (2013).

We have seen that the deferred acceptance algorithm leads to a loss of efficiency compared to the top trading cycle. Starting from this idea, O. Kesten (2010) considers whether it is possible to improve the efficiency of the deferred acceptance algorithm by changing the order of schools based on students' preferences. If we return to our initial example, in the first step of the deferred acceptance algorithm, Student 3 applies to School 1, but this does not bring any benefit to him because in the later steps of the algorithm, he was rejected at this school, whereas for Student 1, who is rejected in the first step, School 1 is the best choice. Thus, Student 3 creates negative externalities to Student 1, without any benefit for himself. This is precisely where O. Kesten (2010) sees an opportunity to improve the efficiency of the deferred acceptance algorithm, by deleting critical schools from the list of preferences of the students who create negative externalities without benefits for themselves.

For the application of this algorithm, it is necessary that the students who violate the matching of others without any benefit for themselves should accept the elimination of the critical schools from the list of their preferences. This algorithm is referred to as the Efficiency-Adjusted Deferred-Acceptance Mechanism (EADAM) and in this procedure, the loss of efficiency as a result of the previously described reason is eliminated. It is obvious that this modified algorithm Pareto-dominates the standard deferred acceptance algorithm. If all the students who create negative externalities consent to the elimination of the critical schools from the list of their preferences, the modified algorithm leads to the allocation that is Pareto-efficient.

In the previous example, School 1 should be deleted from the list of the preferences of Student 3, after which, the deferred acceptance algorithm can be applied. However, after this change, we can see that Student 3 re-creates negative externalities to the other students without any benefit for himself by applying to School 2 because he will be rejected in the next steps of the algorithm in this school. Therefore, it is necessary to delete Schools 1 and 2 from the list of preferences of Student 3. After these changes, which are presented in Table 18 and Table 19, the modified deferred acceptance algorithm can be applied.

Table 18 Priorities

$\succ_{c_1}$	$\succ_{c_2}$	$\succ_{c_3}$	$\succ_{c_4}$
s_4	s_1	s_4	s_2
s_2	s_2	s_3	s_3
s_3	s_3	s_2	s_1
s_1	s_4	s_1	s_4

Source: Author

Table 19 Preferences

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$	$\succ_{s_4}$
c_1	c_4		c_4
c_2	c_3		c_2
c_3	c_2	c_4	c_1
c_4	c_1	c_3	c_3

Source: Author

With these changes in the first step of the deferred acceptance algorithm, the situation (Table 20) is as follows:

Table 20 Increasing efficiency (1)

c_1	c_2	c_3	c_4
s_1			s_2, s_3, s_4

Source: Author

Students 3 and 4 are rejected in School 4 and apply to Schools 3 and 2 in the second step (Table 21).

Table 21 Increasing efficiency (2)

c_1	c_2	c_3	c_4
s_1	s_4	s_3	s_2

Source: Author

This allocation corresponds to the top trading cycle allocation. Accordingly, the modified deferred acceptance algorithm results in a Pareto-efficient allocation.

AN INCREASE IN THE SCHOOL'S RANK ON THE PREFERENCE LIST DUE TO ITS HIGHER QUALITY AND MINORITY STUDENTS

The deferred acceptance algorithm can be analyzed in terms of comparative statics, i.e. with respect to how the matching is modified due to the fact that some schools improve their quality. Due to the improved quality of schools, students should increase the ranking of a particular school on the list of their preferences. The deferred acceptance algorithm respects the improvement of school quality if the school is matched to the student with a higher priority after the school increases its quality. Let us consider the initial example and assume that School 3 improves its quality and that Student 3 puts School 3 at the top of the preference list (Table 22 and Table 23).

Table 22 Priorities

$\succ_{c_1}$	$\succ_{c_2}$	$\succ_{c_3}$	$\succ_{c_4}$
s_4	s_1	s_4	s_2
s_2	s_2	s_3	s_3
s_3	s_3	s_2	s_1
s_1	s_4	s_1	s_4

Source: Author

Table 23 Preferences

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$	$\succ_{s_4}$
c_1	c_4	$\mathbf{c_3}$	c_3
c_2	c_3	c_1	c_4
c_3	c_2	c_2	c_2
c_4	c_1	c_4	c_1

Source: Author

After this change, the application of the deferred acceptance algorithm generates the following matching (Table 24).

Table 24 Increasing school quality

c_1	c_2	c_3	c_4
s_3	s_1	s_4	s_2

Source: Author

In this example, the improvement of the quality of School 3 is respected since it is matched to Student 4 instead of Student 3. However, it is easy to construct an example in which the deferred acceptance algorithm does not take into account the improvement of school quality. In addition, J. W. Hatfield, F. Kojima and Y. Narita (2017) prove that the Boston algorithm and the top trading cycle do not always respect the improvement of school quality.

Analyzing this problem of comparative statics in large markets, J. W. Hatfield, F. Kojima and Y. Narita (2017) prove that the deferred acceptance algorithm respects the improvement of school quality in large markets. In other words, after the improvement of school quality, as the size of the market increases, the probability that the school is matched to the student with a lower priority decreases. However, the Boston and the top trading cycle algorithms do not have this property in large markets. Accordingly, the deferred acceptance algorithm provides an incentive for schools to improve their quality, whereas the Boston and the top trading cycle algorithms are deprived of this feature.

Students differ according to their financial situation, social group, race, etc. The schools that are popular are located in the central parts of the city inhabited by wealthy students. Since schools determine their priorities based on the distance of the student's residence from school, the students who are not wealthy do not have a great opportunity to be enrolled in a popular school. For this reason, quotas are introduced for minority students in popular schools. In most cases, this policy brings minority students into a better position. However, F. Kojima (2012) argues that quotas may, in certain cases, make minority students worse off since majority students apply to other popular schools in which there is no quota, thereby reducing the possibility for minority students to enroll.

The following example illustrates the situation when the introduction of quotas makes minority students worse off. In this case, there are 3 students and 2 schools; School 1 has one place and School 2 has 2 places. The students' preferences and the school's priorities are shown in Table 25 and Table 26. Students 1 and 2 are majority students and Student 3 is a minority student.

Table 25 Preferences

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$
c_2	c_2	c_1
c_1	c_1	c_2

Source: Author

Table 26 Priorities

$\succ_{c_1}$	$\succ_{c_2}$
s_1	s_3
s_2	s_2
s_3	s_1
$q_1 = 1$	$q_2 = 2$

Source: Author

First, we will determine the allocation by assuming that there is no quota for minority students. The deferred acceptance algorithm, in which students make a proposal, generates the following allocation (Table 27):

Table 27 Minority students (1)

c_1	c_2
s_3	s_1, s_2

Source: Author

Now suppose that School 2 introduces a quota and reserves one place for the minority student 3. The deferred acceptance algorithm results in the following allocation (Table 28):

Table 28 Minority students (2)

c_1	c_2
s_1	s_2, s_3

Source: Author

After the introduction of the quota, the minority student 3 is matched to the less preferred School 2.

In addition to introducing a quota, another way to favor minority students is to change the priority so that minority students are given a higher priority than majority students, while the priorities within each of these groups stay the same. F. Kojima (2012) proves that the top trading cycle can also make minority students worse off if a quota is introduced for these students or if priorities are changed in their favor.

To reduce the problem that occurs when minority students are made worse off by the introduction of quotas, I. Hafalir, B. Yenmez and M. Yildrim (2013) suggest the use of flexible quotas instead of fixed quotas. When a fixed quota is applied, the school is unable to enroll majority students in the quota for the minority ones, even though there is no sufficient number of minority students to fill the quota. With a flexible quota, the school first enrolls minority students within their quota, whereas the empty places within this quota can be filled with majority students. The simulation analysis carried out by the authors shows that the number of minority students that are better off in the deferred acceptance algorithm and the top trading cycle with a flexible quota is significantly greater than the number of minority students, who are in a better position in the algorithms with a fixed quota.

THE LIMITED LIST OF PREFERENCES

In the previous discussion, we have assumed that students can submit the list of their preferences of an unlimited length. In reality, students have a limit on the length of the list of preferences. In New York, for example, choice is limited to maximum twelve schools, whereas in Boston, it was impossible to specify up to five schools before 2006. With this assumption, it is no longer certain that the deferred acceptance algorithm and the top trading cycle will be incentive-compatible. In other words, in these algorithms, with the limited length of the list of preferences, it is necessary to determine the Nash equilibrium, as well as in the Boston algorithm, without this limitation. Therefore, G. Haeringer and F. Klijn (2009) determine the Nash equilibrium in the algorithms with a limited list of preferences in the Boston mechanism, the deferred acceptance algorithm and the top trading cycle. An important result of this paper is that the Nash equilibria in the Boston algorithm and in the top trading cycle are independent of the length of preferences. On the other hand, the Nash equilibria in the deferred acceptance algorithm have a hierarchical relationship, which means that the Nash equilibrium in the algorithm with a shorter list of preferences is the Nash equilibrium in the algorithm with a longer list of preferences.

The determination of the Nash equilibrium in the deferred acceptance algorithm can be illustrated by an example taken from G. Haeringer and F. Klijn (2009). In this example, there are 3 students and 3 schools, and each school can accept one student. The length of the list of preferences is limited to 2 schools. Table 29 shows the preferences of the students with an unlimited length of the list of preferences, the preferences of a limited length, as well as the priorities of the schools.

By applying the deferred acceptance algorithm, in which students make a proposal, on the basis of the preferences of a limited length, the allocation: $[(s_1, c_1), (s_2, c_2), (s_3, c_3)]$ is obtained. In this allocation, there is justified envy since Student 2 prefers School 3 and has a higher priority in that school than Student 3. Here, the result obtained differs from the earlier conclusions because the Nash equilibrium allocation in the deferred acceptance algorithm does not have to be stable when the length of the preference list is limited.

Table 29 Limited list of preferences

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$	$\succ_{s_1}^{(2)}$	$\succ_{s_2}^{(2)}$	$\succ_{s_3}^{(2)}$	$\succ_{c_1}$	$\succ_{c_2}$	$\succ_{c_3}$
c_1	c_3	c_3	c_1	c_1	c_3	s_3	s_3	s_1
c_2	c_1	c_2	c_3	c_2	c_1	s_1	s_1	s_2
c_3	c_2	c_1				s_2	s_2	s_3

Source: Haeringer & Klijn, 2009

In order to obtain a stable matching in the deferred acceptance algorithm with a limited length of preferences, school priorities must satisfy F. Ergin's acyclicity condition (2002). In the top trading cycle, a stable allocation is not achieved even when there is no limit on the length of the list of preferences. Therefore, in this case, school priorities must satisfy a stricter condition called O. Kesten's acyclicity condition (2006). These conditions include two sub-conditions. The cyclic condition is based on the fact that school priorities form a cycle, such that for example Student 1 has a higher priority in School 1 than Student 3, and Student 3 has a higher priority than Student 1 in School 2. If both schools have the same priorities, the cyclic condition is never fulfilled. The second sub-condition is rarity, which implies that there are a significant number of students applying for places in schools. If each school has the number of seats equal to the number of students, the rarity condition is never met. As the number of places in a school decreases compared to the total number of students, competition for available places is more intensive.

For the top trading cycle allocation to be efficient with the limited length of the list of preferences, it is necessary that school priorities satisfy the *X*-acyclicity, while the efficiency of the deferred acceptance algorithm needs a stricter requirement for school priorities called the strong *X*-acyclicity.

INDIFERENCES IN SCHOOL CHOICE

In the previous discussion, we have assumed that schools have strict priorities when ranking students. In reality, however, students belong to priority groups and schools are indifferent between students within the same group, whereas there is a strict priority between different groups. Matching algorithms cannot be applied in the case when priorities are not strict and it is necessary to transform weak priorities into strict priorities. One option for the indifference problem suggested by A. Erdil and H. Ergin (2008) is that the students who have a lower index within the same priority group have a higher priority. For example, if Students 1, 2 and 3 belong to the same group, Student 1 has the highest priority and is followed by Student 2 and student 3. Such an arbitrary rule does not guarantee that the allocation in the deferred acceptance algorithm is stable. Therefore, A. Erdil and H. Ergin (2008) propose a stable improvement cycle in order to transform an arbitrary matching into a stable matching.

In addition to the previous option, for the indifference problem in school choice, A. Abdulkardiroğlu, P. Pathak and A. Roth (2009) propose single and multiple tie-breaking rules for the resolution of indifference. In Multiple tie-breaking (DA-MTB), each student is assigned a different lottery number in each school, whereas in Single tie-breaking (DA-STB), each student is assigned the same lottery number in each school. It is possible to prove that the average ranking of the schools which students are enrolled in is higher on their list of preferences in the DA-STB than in the DA-MTB.

All of the previous methods have in common that students do not have any influence on the generation of strict priorities from weak priorities. This problem can further be improved by applying the deferred acceptance algorithm, in which students have the opportunity to influence the resolution of indifference. This algorithm was constructed by A. Abdulkadiroğlu, Y-K. Che and Y. Yasuda (2015) and it is referred to as Choice-Augmented Deferred Acceptance (CADA). A simplified explanation for this algorithm can be illustrated by the following example, in which there are three students and three schools, and each school can only enroll one student. All the students belong to the same priority group, which means that the schools are indifferent between them. The students have the following cardinal utilities for different schools (Table 30).

Table 30 Cardinal utilities and school choice

	$u(s_1)$	$u(s_2)$	$u(s_3)$
c_1	4	4	3
c_2	1	1	2
c_3	0	0	0

*Source***:** Abdulkardiroğlu, Che & Yasuda, 2015

First, we will determine the allocation in the deferred acceptance algorithm, in which indifference is resolved such that each student receives a lottery number from a uniform distribution. By generating strict priorities in this way, each student has the same probability of 1/3 to enroll in any school, so that each student has the expected utility of 5/3. However, a Pareto improvement is possible in this case. Student 3 has a higher level of utility if he is enrolled in School 2, which is his second best choice compared to Students 1 and 2, and the matching in which Student 3 is certainly enrolled in School 2, whereas Students 1 and 2 are enrolled in Schools 1 and 3, with the probability of 1/2, is Pareto-superior to the initial situation, when all students participate in the lottery. In the latter case, each student has an expected utility of 2, which is higher than the level of the expected utility of 5/3. In order to obtain this matching, each student needs to be offered a choice between certain enrolment in School 2 and the lottery, in which he is enrolled in School 1 and School 3, with an equal probability. Students 1 and 2 will choose the lottery, and Student 3 will choose safe enrolment in School 2.

The matching that we have previously described can be achieved with the CADA algorithm. In this algorithm, students submit a list of their preferences and one target school. In the resolution of indifference in a particular school, the students who have indicated that school as the target have a priority. Each student receives two lottery numbers drawn from a uniform distribution. The first lottery number that the student receives is the target lottery number, and the second is the regular lottery number. In the determination of strict priorities, the target lottery number is first considered, and then the regular lottery number is considered. Once indifferences have been resolved, the deferred acceptance algorithm is applied.

Generating strict priorities based on the target and the regular lottery numbers can be illustrated by way of the example in which there are ten students and two schools. Students: 1, 3, 5, 7, and 9 target School 1 and Students: 2, 4, 6, 8, and 10 target School 2. Suppose that the students obtain the following target: *T(I)*, and regular: *R(I)*, lottery numbers:

T(I): 7, 1, 2, 8, 3, 4, 9, 5, 6, 10; R(I): 7, 2, 4, 3, 5, 8, 9, 6, 10, 1.

For the students with the odd index, who have targeted School 1, the priority is determined based on the target lottery number, and the priority of the students in the first school is: 7, 1, 3, 9, 5. After that, the priority of the students who have not targeted School 1 is determined on the basis of the regular lottery number, so that the complete order of the priorities in the first school is: 7, 1, 3, 9, 5, 2, 4, 8, 6, 10. The second school is targeted by the students with the even index and based on the target lottery number, the priority for these students is: 2, 8, 4, 6, 10. The priority for other students is determined and it is based on the regular lottery number, thus the complete priority order in the second school being: 2, 8, 4, 6, 10, 7, 3, 5, 9, 1.

The simulation analysis conducted by A. Abdulkadiroğlu, Y-K. Che and Y. Yasuda (2015) shows that in the deferred acceptance algorithm with the multiple tie-breaking rule, a smaller number of students are enrolled in the first choice schools compared with the deferred acceptance algorithm with the single tie-breaking rule and the deferred acceptance algorithm with a choice. As regards the last two algorithms, when the number of the students enrolled in the first choice schools is compared, there is no significant difference between them. However, the deferred acceptance algorithm with a choice has an advantage over the deferred acceptance algorithm with the single tie-breaking rule for the students enrolled in the school that is their *k*-th best choice, because these students have a higher utility in the CADA algorithm.

CONCLUSION

In this paper, we have presented the most important results in matching students to schools by using the simplified examples, thus making this field closer to a broader audience. We have seen that there are some limiting factors in the application of the matching algorithms. First, the deferred acceptance algorithm is not efficient, which is its main drawback. The second limiting factor in the application of the matching algorithms is the limit of the preference list that students can submit, which undermines the stability of matching. From the practical point of view, students should have a possibility of submitting a sufficiently long list of schools so that the length of the preference list is not a limiting factor. In reality, the vast majority of students submit their preferences for several schools, which implies that the limit on the length of the preference list is not so important in practical application.

We have shown in the paper that cooperative game theory is possible to apply in matching problems. Moreover, we have seen that it is possible to make incentive-compatible mechanisms where students reveal their preferences.

The previous conclusions are implicative of the fact that the relationship between the theoretical models and practice is bidirectional. The absence of the price mechanism has imposed the need for the creation of the alternative rules as a substitute for the market. On the other hand, the existing theoretical knowledge in cooperative game theory and the mechanism design has enabled the achievement of this objective and the finding out of a solution to the practical problem.

The matching algorithms have proved to be very successful in determining the optimal allocation in the situations in which the market mechanism cannot be used for legal or ethical reasons. Beside their application to the matching of students to schools, the algorithms have been successfully applied to the matching of doctors to hospitals, matching organ donors to patients, the allocation of parking spaces or offices, and so on. In addition to matching students to schools, their especially significant application is that in matching organ donors to patients, where the incompatibility problem is greatly reduced.

Matching students to schools is of great importance in the Republic of Serbia since the algorithm of immediate matching is still used and this paper proposes the improvements that could be achieved by applying the deferred acceptance algorithm.

It would be interesting for further research to analyze an increase in students' welfare if the deferred acceptance algorithm is used instead of the immediate matching algorithm. However, based on the historical data, the limiting factor in this analysis is that only information on the stated preferences is available and it is known that the immediate matching algorithm does not induce a true preference revelation. Therefore, based on the historical data, it is possible to determine an increase in welfare only for the stated preferences, i.e. an increase in welfare inclusive of this constraint.

REFERENCES

Abdulkadiroğlu, A. (2013). School choice. In N. Vulkan, A. E. Roth & Z. Neeman (Eds.), *The Handbook of Market Design* (pp. 138-169). Oxford, UK: Oxford University Press.

Abdulkadiroğlu, A., Che, Y-K., & Yasuda, Y. (2015). Expanding "choice" in school choice. *American Economic Journal: Microeconomics*, *7*(1), 1-42. doi: 10.1257/mic.20120027

Abdulkardiroğlu, A., & Sönmez, T. (2003). Social choice: A mechanism design approach. *American Economic Review*, *93*(3), 729-747. doi: 10.1257/000282803322157061

Abdulkardiroğlu, A., Pathak, P., & Roth, A. (2005*a*). The New York city high school match. *American Economic Review*, *95*(2), 364-367. doi: 10.1257/000282805774670167

Abdulkardiroğlu, A., Pathak, P., Roth A., & Sönmez, T. (2005*b*). The Boston public school match. *American Economic Review*, *95*(2), 368-371. doi: 10.1257/000282805774669637

Abdulkardiroğlu, A., Pathak, P., & Roth, A. (2009). Strategy-proofness *versus* efficiency in school choice with indifferences: Redesigning the NYC school match. *American Economic Review*, *99*(5), 1954-1978. doi: 10.1257/aer.99.5.1954

Backović, M. i Popović, Z. (2012). *Matematičko modeliranje i optimizacija*. Beograd, Republika Srbija: Ekonomski fakultet Univerziteta u Beogradu.

Backović, M., Popović, Z., & Stamenković, M. (2016). Reflexive game theory approach to mutual insurance problem. *Montenegrin Journal of Economics*, *12*(3), 87-100. doi: 10.14254/1800-5845.2016/12-3/6

Erdil, A., & Ergin, H. (2008). What's the matter with tie-breaking? Improving efficiency in school choice. *American Economic Review*, *98*(3), 669-689. doi: 10.1257/aer.98.3.669

Ergin, H., & Sönmez, T. (2006). Games of school choice under Boston mechanism. *Journal of Public Economics*, *90*(1-2), 215-237. doi:10.1016/j.jpubeco.2005.02.002

Ergin, H. (2002). Efficient resource allocation on the basis of priorities. *Econometrica*, *70*(6), 2489-2497. doi: 10.1111/j.1468-0262.2002.00447.x

Gale, D., & Shapely, L. (1962). College admission and the stability of marriage. *American Mathematical Monthly*, *69*(1), 9-15.

Gale, D., & Scarf, H. (1974). On cores and indivisibility. *Journal of Mathematical Economics*, 1, 23-37.

Haeringer, G., & Klijn, F. (2009). Constrained school choice. *Journal of Economic Theory*, *144*(5), 1921-1947. doi.org/10.1016/j.jet.2009.05.002

Hafalir, I., Yenmez B., & Yildirim, M. (2013). Effective Affirmative Action in School Choice. *Theoretical Economics 8*(2), 325-363. doi: 10.3982/te1135

Hatfield, J. W., Kojima, F., & Narita, Y. (2017). Promoting school competition through school choice: A market design approach. *Journal of Economic Theory.* (Forthcoming)

Kesten, O. (2006). On two competing mechanisms for priority based allocation problems. *Journal of Economic Theory*, *127*, 155-171. doi: 10.1016/j.jet.2004.11.001

Kesten, O. (2010). School choice with consent. *Quarterly Journal of Economics*, *125*(3), 1297-1348. doi: https://doi.org/10.1162/qjec.2010.125.3.1297

Kojima, F. (2012). School choice: Impossibilities for affirmative action. *Games and Economic Behavior*, *75*(2), 685-693. doi.org/10.1016/j.geb.2012.03.003

Kojima, F., & Pathak, P. (2009). Incentives and stability in large two-sided matching markets. *American Economic Review*, *99*(3), 608-627. doi: 10.1257/aer.99.3.608

Kojima, F., & Ünver, U. (2014). The "Boston" school-choice mechanism: An axiomatic approach. *Economic Theory*, *55*(3), 515-544. doi:10.1007/s00199-013-0769-8

Pathak, P., & Sönmez, T. (2008). Leveling the playing field: Sincere and sophisticated players in the the Boston mechanism. *American Economic Review*, *98*(4), 1636-1652. doi: 10.1257/aer.98.4.1636

Roth, A. (1985). The college admission problem is not equivalent to the marriage problem. *Journal of Economic Theory*, *36*(2), 277-288. doi: 10.1016/0022-0531(85)90106-1

Roth, A. (2015). *Who Gets What - And Why*. New York, NY: Houghton Mifflin Harcourt Publishing Company.

Sönmez, T. (1997). Manipulation *via* capacities in two-sided matching markets. *Journal of Economic Theory*, *77*(1), 197-204. doi.org/10.1006/jeth.1997.2316

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